

# SOME ESTIMATES FOR ROUGH MULTILINEAR FRACTIONAL INTEGRAL OPERATORS AND ROUGH MULTI-SUBLINEAR FRACTIONAL MAXIMAL OPERATORS

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ABSTRACT. The aim of this paper is to get the  $L^p$ -estimates, weighted estimates and two-weighted estimates for rough multilinear fractional integral operators and rough multi-sublinear fractional maximal operators, respectively.

## 1. INTRODUCTION

It is well known that, for the purpose of researching non-smoothness partial differential equation, mathematicians pay more attention to the singular integrals with rough kernel. Moreover, the fractional type operators and their weighted boundedness theory play important roles in harmonic analysis and other fields, and the multilinear operators arise in numerous situations involving product-like operations, see [6, 7, 8, 10, 11] for instance.

Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space of points  $x = (x_1, \dots, x_n)$  with norm  $|x| = \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$  and  $(\mathbb{R}^n)^m = \mathbb{R}^n \times \dots \times \mathbb{R}^n$  be the  $m$ -fold product spaces ( $m \in \mathbb{N}$ ). Throughout this paper, we denote by  $\vec{y} = (y_1, \dots, y_m)$  and  $x, y_1, \dots, y_m \in \mathbb{R}^n$ ,  $d\vec{y} = dy_1 \dots dy_m$ , and by  $\vec{f}$  the  $m$ -tuple  $(f_1, \dots, f_m)$ ,  $m, n$  the nonnegative integers with  $n \geq 2, m \geq 1$ . Let also  $S^{mn-1}$  denote the unit sphere of  $\mathbb{R}^{mn}$  only with the condition  $\Omega \in L^s(S^{mn-1})$  for some  $s > 1$ .

Motivated by [6, 7, 8, 10], in this paper we will introduce the following rough multilinear fractional operators  $I_{\Omega, \alpha}^{(m)}$  and rough multi-sublinear fractional maximal operators  $M_{\Omega, \alpha}^{(m)}$ , which are the more generalizations of the classical setting and study them on product spaces  $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$

$$I_{\Omega, \alpha}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m f_i(x - y_i) d\vec{y},$$

$$M_{\Omega, \alpha}^{(m)}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y}.$$

In case of  $m = 1$ , we denote  $M_{\Omega, \alpha}^{(1)}$  and  $I_{\Omega, \alpha}^{(1)}$  by  $M_{\Omega, \alpha}$  and  $I_{\Omega, \alpha}$ , which are the rough sublinear fractional maximal operator and the rough fractional integral operator mentioned in [1, 3, 4, 9], respectively. In the case of  $\Omega = 1$ , we denote

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$M_{1,\alpha}^{(1)}$  and  $I_{1,\alpha}^{(1)}$  by  $M_\alpha^{(m)}$  and  $I_\alpha^{(m)}$ , which are the multi-sublinear fractional maximal operator and the multilinear fractional integral operator, respectively. If  $\Omega = 1$  and  $m = 1$ , then they become the classical fractional maximal operator  $M_\alpha$  and the classical fractional integral operator (Riesz potential)  $I_\alpha$ .

In 1971, Muckenhoupt and Wheeden [14] studied the weighted norm inequalities for  $I_{\Omega,\alpha}$  with the power weight; and they [15] showed in 1974 that  $\|I_\alpha f\|_{L_{w^q}^q} \leq C \|f\|_{L_{w^p}^p}$  and  $\|M_\alpha f\|_{L_{w^q}^q} \leq C \|f\|_{L_{w^p}^p}$ , where  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$  and  $w \in A_{p,q}$ . A locally integrable nonnegative function  $w$  on  $\mathbb{R}^n$  is said to belong to  $A_{p,q}$  ( $1 < p, q < \infty$ ) if

$$\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q w(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty,$$

where  $Q$  denotes a  $n$ -dimensional cube in  $\mathbb{R}^n$  with the sides parallel to the coordinate axes and the supremum is taken over all cubes, and as usual,  $p'$  is the exponent conjugate to  $p$  satisfying  $\frac{1}{p} + \frac{1}{p'} = 1$ .

In 1993, Chanillo, Watson and Wheeden [2] proved that the operator  $I_{\Omega,\alpha}$  is of weak type  $\left(1, \frac{n}{n-\alpha}\right)$  when  $\Omega \in L^s(S^{n-1})$  ( $s \geq \frac{n}{n-\alpha}$ ). In 1998, Ding and Lu [3] extended Muckenhoupt-Wheeden's result to get that, when  $s' < p < \frac{n}{\alpha}$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ ,

$$\begin{aligned} \left( \int_{\mathbb{R}^n} |I_{\Omega,\alpha} f(x) w(x)|^q dx \right)^{\frac{1}{q}} &\leq C \left( \int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{\frac{1}{p}}, \\ \left( \int_{\mathbb{R}^n} |M_{\Omega,\alpha} f(x) w(x)|^q dx \right)^{\frac{1}{q}} &\leq C \left( \int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

for  $\Omega \in L^s(S^{n-1})$  and  $w(x)^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}$ ,  $\frac{1}{s} + \frac{1}{s'} = 1$ . In 2001, Ding and Lin [4] proved some two-weighted norm inequalities for  $I_{\Omega,\alpha}$ . García-Cuerva and Martell [5] also showed in 2001 that, when  $0 < \alpha < n$ ,  $1 < p < q < \infty$ ,

$$\begin{aligned} \left( \int_{\mathbb{R}^n} |I_\alpha f(x)|^q u(x) dx \right)^{\frac{1}{q}} &\leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{\frac{1}{p}}, \\ \left( \int_{\mathbb{R}^n} |M_\alpha f(x)|^q u(x) dx \right)^{\frac{1}{q}} &\leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{\frac{1}{p}}, \end{aligned}$$

for weights  $(u, v)$ , if there exists  $r > 1$  for each cube  $Q$  in  $\mathbb{R}^n$ , such that

$$|Q|^{\frac{1}{q} + \frac{\alpha}{n} - \frac{1}{p}} \left( \frac{1}{|Q|} \int_Q u(x)^r dx \right)^{\frac{1}{rq}} \left( \frac{1}{|Q|} \int_Q v(x)^{r(1-p')} dx \right)^{\frac{1}{rp'}} \leq C.$$

The multilinear operators are natural generalizations of linear case. In recent years, many authors have been interested in these topics. [6, 7, 8, 10] are some

important papers on multilinear operators. In 1992, Grafakos [6] first studied the multilinear maximal function and multilinear fractional integral defined by

$$M_{\alpha}^{(m)}(\vec{f})(x) = \sup_{t>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} \left| \prod_{i=1}^m f_i(x - \theta_i y) \right| dy$$

and

$$I_{\alpha}^{(m)}(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^m f_i(x - \theta_i y) dy,$$

where  $\theta_i (i = 1, \dots, m)$  are fixed distinct nonzero real numbers and  $0 < \alpha < n$ . We note that, if we simply take  $m = 1$  and  $\theta_i = 1$ , then  $M_{\alpha}$  and  $I_{\alpha}$  are just the operators studied by Muckenhoupt and Wheeden in [15]. In 1999, Kenig and Stein [10] considered another type of multilinear fractional integral which was defined by

$$I_{\alpha,A}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m f_i(\ell_i(y_1, \dots, y_m, x)) d\vec{y},$$

where  $\ell_i$  is a linear combination of  $y_j (j = 1, \dots, m)$  and  $x$  depending on the matrix  $A$ . They have proved that  $I_{\alpha,A}$  is of strong type  $(L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}, L^q)$  and weak type  $(L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}, L^{q,\infty})$ . Hence, it will be an interesting question whether Kenig and Stein's famous result (Theorem 1 in [10]) and above weighted conclusions can be extended to the operators  $I_{\Omega,\alpha}^{(m)}$  and  $M_{\Omega,\alpha}^{(m)}$  for  $m > 1$  and non-smooth kernel  $\Omega$ .

In this paper, we will give the positive answers to this question, and simultaneity extend Kenig and Stein's famous result (Theorem 1 in [10]) to the context for rough multilinear fractional integral operators  $I_{\Omega,\alpha}^{(m)}$  and rough multi-sublinear fractional maximal operators  $M_{\Omega,\alpha}^{(m)}$ , and also show their weighted boundedness, respectively. Notice that, in special case, we can also obtain the weighted boundedness of the operators  $I_{\alpha}^{(m)}$  and  $M_{\alpha}^{(m)}$ , respectively.

Throughout this paper, the letter  $C$  always remains to denote a positive constant that may varies at each occurrence but is independent of the essential variable.

## 2. DEFINITIONS AND MAIN RESULTS

**Definition 1. (Multilinear fractional type operators)** Let  $\vec{f} = (f_1, \dots, f_m)$  and  $0 < \alpha < mn$ .

(i) Suppose each  $f_i (i = 1, \dots, m)$  is locally integrable on  $\mathbb{R}^n$ . Then for any  $x \in \mathbb{R}^n$ , we define the multi-sublinear fractional maximal operator by

$$M_{\alpha}^{(m)}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y}.$$

Obviously, in the case of  $\alpha = 0$ ,  $M_{\alpha}^{(m)}$  is the multi-sublinear maximal operator  $M^{(m)}$  and also; in the case of  $m = 1$ ,  $M_{\alpha}^{(m)}$  is the classical fractional maximal operator  $M_{\alpha}$ . It's clear that  $M_{\alpha}$ ,  $M^{(m)}$  and  $M_{\alpha}^{(m)}$  are all the special case of  $M_{\Omega,\alpha}^{(m)}$  defined already in first section.

(ii) Define the multilinear fractional integral operator as

$$I_{\alpha}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y},$$

where  $d\vec{y} = dy_1 \dots dy_m$  and  $|\vec{y}| = |y_1| + \dots + |y_m|$ .

**Remark 1.** The above multilinear fractional integral operator has been studied by Kenig and Stein in [10], in which the weak and strong estimates were given for a class of generalized fractional integral operators.

**Definition 2. (Class of  $A_p$  and  $A_{p,q}$ )** A locally integrable nonnegative function  $w$  on  $\mathbb{R}^n$  is said to belong  $A_p$  ( $1 < p < \infty$ ) if there exists  $C$  such that

$$\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C,$$

where  $Q$  denotes a  $n$ -dimensional cube in  $\mathbb{R}^n$  with the sides parallel to the coordinate axes and the supremum is taken over all cubes. When  $p = 1$ , a nonnegative measurable function  $w$  is said to belong to  $A_1$ , if there exists  $C$  such that for any cube  $Q \subset \mathbb{R}^n$ ,

$$\frac{1}{|Q|} \int_Q w(y) dy \leq Cw(x), \quad \text{a.e. } x \in Q.$$

Recall the definition of  $A_{p,q}$  weight defined in first section, one can see that  $w \in A_{p,p}$  if and only if  $w^p \in A_p$  for  $1 < p < \infty$ . Moreover, if  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$  with  $1 < p < \frac{n}{\alpha}$  and  $0 < \alpha < n$ , then it's easy to deduce that

$$w(x) \in A_{p,q} \iff w(x)^q \in A_{\frac{q(n-\alpha)}{n}} \iff w(x)^q \in A_{1+\frac{q}{p}}.$$

**Definition 3. (Class of  $A_{p,q}^{\alpha}$ )** A pair of weights  $(u, v)$  is said to belong to  $A_{p,q}^{\alpha}$ , where  $1 \leq p \leq q < \infty$  and  $0 \leq \alpha < n$ , if there exists a positive constant  $C$  independent of the cube  $Q$  in  $\mathbb{R}^n$  such that

$$|Q|^{\frac{1}{q} + \frac{\alpha}{n} - \frac{1}{p}} \left( \frac{1}{|Q|} \int_Q u(x) dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_Q v(x)^{(1-p')} dx \right)^{\frac{1}{p'}} \leq C,$$

when  $1 < p < \infty$ ; and

$$|Q|^{\frac{1}{q} + \frac{\alpha}{n} - 1} \left( \frac{1}{|Q|} \int_Q u(x) dx \right)^{\frac{1}{q}} \leq Cv(x) \quad \text{a.e. } x \in Q,$$

when  $p = 1$ . Recall the definition of  $A_p$  weight, it is easy to see that  $(u, v) \in A_p$  if and only if  $(u, v) \in A_{p,p}^0$  for  $1 \leq p < \infty$ .

**Theorem 1.** Let  $0 < \alpha < mn$ ,  $1 \leq s' < \frac{mn}{\alpha}$ , and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(S^{mn-1})$  ( $s > 1$ ),  $\frac{1}{s} + \frac{1}{s'} = 1$ . Let  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$ .

(i) If each  $s' < p_i$ , then there exists a constant  $C > 0$  such that

$$(2.1) \quad \left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

(ii) If  $p_i = s'$  for some  $i$ , then there exists a constant  $C > 0$  such that

$$(2.2) \quad \left\| M_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L^{p, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

**Theorem 2.** Suppose the same conditions and notations of that in Theorem 1,

(i) if each  $s' < p_i$ , then there exists a constant  $C > 0$  such that

$$(2.3) \quad \left\| I_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)};$$

(ii) if  $p_i = s'$  for some  $i$ , then there exists a constant  $C > 0$  such that

$$(2.4) \quad \left\| I_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L^{p, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

**Theorem 3.** Let  $0 < \alpha < mn$ ,  $1 \leq s' < \frac{mn}{\alpha}$ , and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(S^{mn-1})$  ( $s > 1$ ),  $\frac{1}{s} + \frac{1}{s'} = 1$ . Suppose that  $f_i \in L_{w^{p_i}}^{p_i}(\mathbb{R}^n)$

with  $s' < p_i < \frac{mn}{\alpha}$  ( $i = 1, 2, \dots, m$ ) and  $w(x)^{s'} \in \bigcap_{i=1}^n A_{s', \frac{q}{s'}}^{\frac{p}{s'}}$ , where  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$ .

If let  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha}{n}$ , then there is a constant  $C > 0$ , independent of  $f_i$ , such that

$$(2.5) \quad \left\| M_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L_{w^p}^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)}.$$

**Theorem 4.** Suppose the same conditions and notations of that in Theorem 3, then there is a constant  $C > 0$ , independent of  $f_i$ , such that

$$(2.6) \quad \left\| I_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L_{w^p}^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)}.$$

**Theorem 5.** Let  $0 < \alpha < mn$ ,  $1 \leq s' < \frac{mn}{\alpha}$ , and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(S^{mn-1})$  ( $s > 1$ ),  $\frac{1}{s} + \frac{1}{s'} = 1$ . Assume that  $(u, v)$  is a pair of weights,  $s' < p_i < q_i < \infty$ , for each  $i = 1, 2, \dots, m$ . Let  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ ,  $\frac{1}{p_i} + \frac{1}{p'_i} = 1$ . If there exists  $r_i > 1$  such that, for every cube  $Q$  in  $\mathbb{R}^n$ ,

$$(2.7) \quad |Q|^{\frac{s'}{q_i} + \frac{\alpha s'}{mn} - \frac{s'}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x) dx \right)^{\frac{s'}{q_i}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1 - (\frac{p_i}{s'})')} dx \right)^{\frac{1}{r_i(\frac{p_i}{s'})'}} \leq C,$$

then for arbitrary  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$ , there is a constant  $C > 0$ , independent of  $f_i$ , such that

$$(2.8) \quad \left\| M_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L_u^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}.$$

**Theorem 6.** Let  $0 < \alpha < mn$ ,  $1 \leq s' < \frac{mn}{\alpha}$ , and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(S^{mn-1})$  ( $s > 1$ ),  $\frac{1}{s} + \frac{1}{s'} = 1$ ,  $(u, v)$  is a pair of weights. If for

every  $i = 1, 2, \dots, m$ ,  $s' < p_i < mp < \infty$  and there exists  $r_i > 1$  such that for every cube  $Q$  in  $\mathbb{R}^n$ , such that

$$|Q|^{\frac{s'}{mp} + \frac{\alpha s'}{mn} - \frac{s'}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x)^{r_i} dx \right)^{\frac{s'}{r_i mp}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1 - (\frac{p_i}{s'})')} dx \right)^{\frac{1}{r_i(\frac{p_i}{s'})'}} \leq C,$$

then for arbitrary  $f_i \in L_v^{p_i}(\mathbb{R}^n)$ , there is a constant  $C > 0$ , independent of  $f_i$ , such that

$$(2.9) \quad \left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_u^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_v^{p_i}(\mathbb{R}^n)}.$$

**Remark 2.** Theorem 2 implies the well-known Hardy-Littlewood-Sobolev Theorem [13], i.e. the case  $m = 1$ ,  $\Omega \equiv 1$  and  $s = \infty$ . Theorem 2 implies Theorem 1 in [10] when  $\Omega \equiv 1$  and  $s = \infty$ . One can also obtain from (2.4) of Theorem 2 that the operator  $I_{\Omega, \alpha}$  is weak type  $\left(1, \frac{n}{n-\alpha}\right)$ . Theorem 4 extends the weighted boundedness of  $I_{\Omega, \alpha}$  in [3]. If  $m = 1$ ,  $\Omega \equiv 1$  and  $s = \infty$ , Theorem 2.6 becomes the results of García-Cuerva and Martell [5], where the two weighted boundedness of  $I_\alpha$  is considered.

### 3. $L^p$ -ESTIMATES FOR ROUGH MULTILINEAR FRACTIONAL INTEGRAL OPERATORS AND ROUGH MULTI-SUBLINEAR FRACTIONAL MAXIMAL OPERATORS

In this section we derive some integral inequalities for multi-sublinear fractional maximal functions  $M_{\Omega, \alpha}^{(m)}(\vec{f})(x)$ , and establish some relationship between  $M_{\Omega, \alpha}^{(m)}$  and  $I_{\Omega, \alpha}^{(m)}$ . We first need some lemmas.

**Lemma 1.** Let  $0 < \alpha < mn$ ,  $1 \leq s' < \frac{mn}{\alpha}$ , and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(S^{mn-1})$  ( $s > 1$ ),  $\frac{1}{s} + \frac{1}{s'} = 1$ , assume that the function  $f_i \in L^{p_i}(\mathbb{R}^n)$  with  $1 \leq p_i \leq \infty$  ( $i = 1, 2, \dots, m$ ), then there exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}^n$ ,

$$(3.1) \quad M_{\Omega, \alpha}^{(m)}(\vec{f})(x) \leq C \left[ M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x) \right]^{\frac{1}{s'}}.$$

*Proof.* By  $s > 1$ ,  $\Omega \in L^s(S^{mn-1})$  and the Hölder's inequality, we have

$$\begin{aligned}
& \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}| < r} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
& \leq \frac{1}{r^{mn-\alpha}} \left( \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \left( \int_{|\vec{y}| < r} |\Omega(\vec{y})|^s d\vec{y} \right)^{\frac{1}{s}} \\
& \leq C \sup_{r>0} \frac{1}{r^{mn(1-\frac{1}{s})-\alpha}} \left( \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
& \leq C \sup_{r>0} \left( \frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
& \leq C \left( \sup_{r>0} \frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
& \leq C \left[ M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x) \right]^{\frac{1}{s'}}.
\end{aligned}$$

This completes the proof of the Lemma 1.  $\square$

**Lemma 2.** Let  $0 < \alpha < mn$ , and let  $f_i \in L^{p_i}(\mathbb{R}^n)$  with  $1 \leq p_i \leq \infty$  ( $i = 1, 2, \dots, m$ ). For any  $0 < \epsilon < \min\{\alpha, mn - \alpha\}$ , there exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}^n$ ,

$$(3.2) \quad \left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right| \leq C \left[ M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})(x) \right]^{\frac{1}{2}} \left[ M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})(x) \right]^{\frac{1}{2}}.$$

*Proof.* Fix  $x \in \mathbb{R}^n$  and  $0 < \epsilon < \min\{\alpha, mn - \alpha\}$ , for any  $\delta > 0$  we decompose as follow,

$$\begin{aligned}
\left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right| & \leq \int_{(\mathbb{R}^n)^m} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
& \leq \int_{|\vec{y}| < \delta} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} + \int_{|\vec{y}| > \delta} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
& =: J_1 + J_2.
\end{aligned}$$

Now, we will estimate  $J_1$  and  $J_2$ , respectively. For  $J_1$ ,

$$\begin{aligned}
J_1 &= \sum_{j=0}^{\infty} \int_{\vec{y} \in B(2^{-j}\delta) \setminus B(2^{-j-1}\delta)} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\delta)^{mn-\alpha}} \int_{\vec{y} \in B(2^{-j}\delta) \setminus B(2^{-j-1}\delta)} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\delta)^{mn-\alpha}} \int_{\vec{y} \in B(2^{-j}\delta)} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq C \sum_{j=0}^{\infty} \frac{(2^{-j}\delta)^{\epsilon}}{(2^{-j}\delta)^{mn-\alpha+\epsilon}} \int_{\vec{y} \in B(2^{-j}\delta)} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq C \delta^{\epsilon} \sum_{j=0}^{\infty} (2^{-j\epsilon}) M_{\Omega, \alpha-\epsilon}^{(m)}(\vec{f})(x) \\
&\leq C \delta^{\epsilon} M_{\Omega, \alpha-\epsilon}^{(m)}(\vec{f})(x).
\end{aligned}$$

As for  $J_2$ , we have

$$\begin{aligned}
J_2 &= \sum_{j=0}^{\infty} \int_{\vec{y} \in B(2^{j+1}\delta) \setminus B(2^j\delta)} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^{mn-\alpha}} \int_{\vec{y} \in B(2^{j+1}\delta) \setminus B(2^j\delta)} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^{mn-\alpha}} \int_{\vec{y} \in B(2^{j+1}\delta)} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq C \sum_{j=0}^{\infty} \frac{(2^j\delta)^{\epsilon}}{(2^j\delta)^{mn-\alpha-\epsilon}} \int_{\vec{y} \in B(2^{j+1}\delta)} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
&\leq C \delta^{-\epsilon} \sum_{j=0}^{\infty} (2^{-j\epsilon}) M_{\Omega, \alpha+\epsilon}^{(m)}(\vec{f})(x) \\
&\leq C \delta^{-\epsilon} M_{\Omega, \alpha+\epsilon}^{(m)}(\vec{f})(x).
\end{aligned}$$

Thus we get

$$\left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right| \leq C \delta^{\epsilon} M_{\Omega, \alpha-\epsilon}^{(m)}(\vec{f})(x) + C \delta^{-\epsilon} M_{\Omega, \alpha+\epsilon}^{(m)}(\vec{f})(x).$$

Now we take  $\delta > 0$  such that

$$\delta^{\epsilon} M_{\Omega, \alpha-\epsilon}^{(m)}(\vec{f})(x) = \delta^{-\epsilon} M_{\Omega, \alpha+\epsilon}^{(m)}(\vec{f})(x).$$

This implies the Lemma 2.  $\square$

We now can prove the  $L^p$  boundedness for operators  $M_{\alpha}^{(m)}$ .



**Theorem 7.** Let  $0 < \alpha < mn$ , and let  $f_i \in L^{p_i}(\mathbb{R}^n)$  with  $1 \leq p_i \leq \infty$  ( $i = 1, 2, \dots, m$ ), and let  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$ .

(i) If each  $p_i > 1$ , then there exists a constant  $C > 0$  such that

$$(3.3) \quad \left\| M_{\alpha}^{(m)}(\vec{f}) \right\|_{L^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

(ii) If  $p_i = 1$  for some  $i$ , then there exists a constant  $C > 0$  such that

$$(3.4) \quad \left\| M_{\alpha}^{(m)}(\vec{f}) \right\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

*Proof.* Fix  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} & \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\ & \leq \int_{|\vec{y}| < r} \frac{1}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\ & \leq \int_{(\mathbb{R}^n)^m} \frac{1}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\ & = I_{\alpha}^{(m)}(|f_1|, |f_2|, \dots, |f_m|)(x). \end{aligned}$$

Taking the supremum for  $r > 0$  on both sides of the above inequality, we have

$$M_{\alpha}^{(m)}(\vec{f})(x) \leq I_{\alpha}^{(m)}(|f_1|, |f_2|, \dots, |f_m|)(x).$$

Applying Theorem 1 in [10] and from the above inequality we immediately obtain the inequalities (3.3) and (3.4).  $\square$

**The proof of Theorem 1.** If each  $s' < p_i$ , we can apply Lemma 1 and Theorem 7 to get

$$\begin{aligned} \left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} \left| M_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} \left| M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x) \right|^{\frac{p}{s'}} dx \right)^{\frac{1}{p}} \\ &\leq C \prod_{i=1}^m \left\| |f_i|^{s'} \right\|_{L^{\frac{p_i}{s'}}(\mathbb{R}^n)}^{\frac{1}{s'}} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

If  $p_i = s'$  for some  $i$ , we also applying Lemma 1 and Theorem 7 to get for any  $\lambda > 0$  that

$$\begin{aligned}
& \left| \left\{ x : \left| M_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) (x) \right| > \lambda \right\} \right| \\
& \leq C \left| \left\{ x : \left| \left[ M_{\alpha s'}^{(m)} \left( |f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'} \right) (x) \right] \right|^{\frac{1}{s'}} > \lambda \right\} \right| \\
& \leq C \left| \left\{ x : \left| \left[ M_{\alpha s'}^{(m)} \left( |f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'} \right) (x) \right] \right| > \lambda^{s'} \right\} \right| \\
& \leq C \left( \frac{1}{\lambda^{s'}} \prod_{i=1}^m \left\| |f_i|^{s'} \right\|_{L^{\frac{p_i}{s'}}(\mathbb{R}^n)} \right)^{\frac{p}{s'}} \leq C \left( \frac{1}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^p.
\end{aligned}$$

Thus, we complete the proof of Theorem 1.

**The proof of Theorem 2.** Take a small positive number  $\epsilon$  with  $0 < \epsilon < \min \left\{ \alpha, \frac{mn}{s'} - \alpha, \frac{n}{p} \right\}$ . One can then see from the conditions of Theorem 2 that  $1 \leq s' < \frac{mn}{\alpha + \epsilon}$  and  $1 \leq s' < \frac{mn}{\alpha - \epsilon}$ , and that

$$(3.5) \quad \frac{1}{q_1} := \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha + \epsilon}{n} = \frac{1}{p} - \frac{\epsilon}{n} > 0,$$

$$(3.6) \quad \frac{1}{q_2} := \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha - \epsilon}{n} = \frac{1}{p} + \frac{\epsilon}{n} > 0.$$

Now if each  $s' < p_i$ , then Theorem 1 implies that

$$\begin{aligned}
\left\| M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) \right\|_{L^{q_1}(\mathbb{R}^n)} & \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}, \\
\left\| M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) \right\|_{L^{q_2}(\mathbb{R}^n)} & \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.
\end{aligned}$$

Note that  $\frac{p}{2q_1} + \frac{p}{2q_2} = 1$ . Using Lemma 2, the Hölder's inequality and the above two inequalities, we obtain

$$\begin{aligned}
\left\| I_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L^p(\mathbb{R}^n)} & = \left( \int_{\mathbb{R}^n} \left| I_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) (x) \right|^p dx \right)^{\frac{1}{p}} \\
& \leq C \left( \int_{\mathbb{R}^n} \left[ M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) (x) \right]^{\frac{p}{2}} \left[ M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) (x) \right]^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\
& = C \left( \int_{\mathbb{R}^n} \left[ M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) (x) \right]^{q_1} dx \right)^{\frac{1}{2q_1}} \left( \int_{\mathbb{R}^n} \left[ M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) (x) \right]^{q_2} dx \right)^{\frac{1}{2q_2}} \\
& = \left\| M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) \right\|_{L^{q_1}(\mathbb{R}^n)}^{\frac{1}{2}} \left\| M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) \right\|_{L^{q_2}(\mathbb{R}^n)}^{\frac{1}{2}} \\
& \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.
\end{aligned}$$

This is the desired inequality (2.3) of Theorem 2.

Similarly, If  $p_i = s'$  for some  $i$ , then Theorem 1 implies that

$$\begin{aligned} \left\| M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) \right\|_{L^{q_1, \infty}(\mathbb{R}^n)} &\leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}, \\ \left\| M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) \right\|_{L^{q_2, \infty}(\mathbb{R}^n)} &\leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

For any  $\lambda > 0$ , we take  $w^2 = \lambda^{\frac{2q_1}{q_1+q_2}} \left( \prod_{i=1}^m \|f_i\|_{L^{p_i}} \right)^{\frac{q_2-q_1}{q_2+q_1}}$ , then Lemma 2 and the above two inequalities give that

$$\begin{aligned} &\left| \left\{ x : \left| I_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) (x) \right| > \lambda \right\} \right| \\ &\leq \left| \left\{ x : \left| M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) (x) \right|^{\frac{1}{2}} > \frac{w}{C} \right\} \right| + \left| \left\{ x : \left| M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) (x) \right|^{\frac{1}{2}} > \frac{\lambda}{w} \right\} \right| \\ &\leq \left| \left\{ x : \left| M_{\Omega, \alpha - \epsilon}^{(m)} \left( \vec{f} \right) (x) \right|^{\frac{1}{2}} > \frac{w^2}{C^2} \right\} \right| + \left| \left\{ x : \left| M_{\Omega, \alpha + \epsilon}^{(m)} \left( \vec{f} \right) (x) \right|^{\frac{1}{2}} > \frac{\lambda^2}{w^2} \right\} \right| \\ &\leq C \left( \frac{1}{w^2} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^{q_2} + C \left( \frac{w^2}{\lambda^2} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^{q_1} \\ &\leq C \left( \frac{1}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^p. \end{aligned}$$

This yields the desired inequality (2.4) of Theorem 2. Thus we complete the proof of Theorem 2.

#### 4. WEIGHTED ESTIMATES FOR ROUGH MULTILINEAR FRACTIONAL INTEGRAL OPERATORS AND ROUGH MULTI-SUBLINEAR FRACTIONAL MAXIMAL OPERATORS

For the proofs of Theorems 3 and 4, we need some lemmas.

**Lemma 3.** [3] Suppose that  $0 < \alpha < n$ ,  $1 \leq s' < p < \frac{n}{\alpha}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$  and  $w(x)^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}$ . Then there exists a small positive number  $\epsilon$  with  $0 < \epsilon < \min \left\{ \alpha, \frac{n}{p} - \alpha, \frac{n}{q'} \right\}$  such that  $w(x)^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}$  and  $w(x)^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}$ , where  $\frac{1}{q_\epsilon} = \frac{1}{p} - \frac{\alpha + \epsilon}{n}$  and  $\frac{1}{q_\epsilon} = \frac{1}{p} - \frac{\alpha - \epsilon}{n}$ .

**Lemma 4.** [15] Assume that  $0 < \alpha < n$ ,  $1 < p < \frac{n}{\alpha}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$  and  $w \in A_{p, q}$ . Then there exists a constant  $C$  independent of  $f$  such that

$$\left( \int_{\mathbb{R}^n} |M_\alpha f(x) w(x)|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{\frac{1}{p}}.$$

**Lemma 5.** Assume that  $0 < \alpha < mn$ , and that  $f_i \in L_{loc}(\mathbb{R}^n)$ ,  $i = 1, 2, \dots, m$ . Then

$$(4.1) \quad M_\alpha^{(m)} \left( \vec{f} \right) (x) \leq C \prod_{i=1}^m M_{\frac{\alpha}{m}} f_i(x).$$

*Proof.* By the definition of  $M_\alpha^{(m)}$  and  $M_\alpha$ , we have

$$\begin{aligned}
& \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \\
& \leq C \frac{1}{r^{mn-\alpha}} \int_{|y_1| < r} \cdots \int_{|y_m| < r} \prod_{i=1}^m |f_i(x - y_i)| dy_1 \dots dy_m \\
& \leq C \prod_{i=1}^m \left( \frac{1}{r^{n-\frac{\alpha}{m}}} \int_{|y_i| < r} |f_i(x - y_i)| dy_i \right) \\
& \leq C \prod_{i=1}^m M_{\frac{\alpha}{m}} f_i(x).
\end{aligned}$$

Taking the supremum for  $r > 0$  on both sides of the above inequality, we obtain the inequality (4.1).  $\square$

Now we can prove the one-weighted boundedness for the operators  $M_\alpha^{(m)}$ .

**Lemma 6.** *Let  $0 < \alpha < mn$ ,  $1 < p_i < \frac{mn}{\alpha}$ ,  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$ ,  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$ ,  $i = 1, \dots, m$ . Assume that  $f_i \in L_{w^{p_i}}^{p_i}(\mathbb{R}^n)$  with weight  $w(x) \in \bigcap_{i=1}^m A_{p_i, q_i}$ , then*

$$\left\| M_\alpha^{(m)}(\vec{f}) \right\|_{L_{w^p}^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)},$$

with the absolute constant  $C$ .

*Proof.* Since  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$ ,  $w(x) \in \bigcap_{i=1}^m A_{p_i, q_i}$ , by the Hölder's inequality, Lemma 4 and Lemma 5, we have

$$\begin{aligned}
\left\| M_\alpha^{(m)}(\vec{f}) \right\|_{L_{w^p}^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} \left| M_\alpha^{(m)}(\vec{f})(x) w(x) \right|^p dx \right)^{\frac{1}{p}} \\
&\leq C \left( \int_{\mathbb{R}^n} \left| \prod_{i=1}^m M_{\frac{\alpha}{m}} f_i(x) w(x) \right|^p dx \right)^{\frac{1}{p}} \\
&\leq C \prod_{i=1}^m \left( \int_{\mathbb{R}^n} \left| M_{\frac{\alpha}{m}} f_i(x) w(x) \right|^{q_i} dx \right)^{\frac{1}{q_i}} \\
&\leq C \prod_{i=1}^m \left( \int_{\mathbb{R}^n} |f_i(x) w(x)|^{p_i} dx \right)^{\frac{1}{p_i}}.
\end{aligned}$$

This proves Lemma 6.  $\square$

**The proof of Theorem 3.** It's easy to see that  $\frac{1}{s'} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha s'}{n} > 0$ ,  $\frac{1}{s'} = \frac{1}{p_i} - \frac{\alpha s'}{mn}$  and  $1 < \frac{p_i}{s'} < \frac{mn}{\alpha s'}$ . Therefore, by Lemma 1 and Lemma 6 we get

$$\begin{aligned} \|M_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_{wP}^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} |M_{\Omega, \alpha}^{(m)}(\vec{f})(x) w(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} |M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x) w(x)^{s'}|^{\frac{p}{s'}} dx \right)^{\frac{1}{p}} \\ &\leq C \prod_{i=1}^m \| |f_i|^{s'} \|_{L_{w^{p_i}}^{\frac{p_i}{s'}}(\mathbb{R}^n)}^{\frac{1}{s'}} \leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

This proves Theorem 3.

**The proof of Theorem 4.** Since  $0 < \alpha < mn$ ,  $s' < p_i < \frac{mn}{\alpha}$ ,  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$  and  $w(x)^{s'} \in \bigcap_{i=1}^n A_{\frac{p_i}{s'}, \frac{q_i}{s'}}$ , we get by Lemma 3 that there exists a small positive number  $\epsilon$  such that  $w(x)^{s'} \in \bigcap_{i=1}^n A_{\frac{p_i}{s'}, \frac{\gamma_i}{s'}}$  and  $w(x)^{s'} \in \bigcap_{i=1}^n A_{\frac{p_i}{s'}, \frac{\xi_i}{s'}}$ , where  $\frac{1}{\gamma_i} = \frac{1}{p_i} - \frac{\alpha + \epsilon}{mn}$ ,  $\frac{1}{\xi_i} = \frac{1}{p_i} - \frac{\alpha - \epsilon}{mn}$ . Put

$$\begin{aligned} \frac{1}{\beta_1} &:= \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha + \epsilon}{n} = \frac{1}{p} - \frac{\epsilon}{n} > 0, \\ \frac{1}{\beta_2} &:= \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha - \epsilon}{n} = \frac{1}{p} + \frac{\epsilon}{n} > 0. \end{aligned}$$

Then, by Theorem 3, we get

$$\begin{aligned} \|M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})\|_{L_{w^{\beta_1}}^{\beta_1}(\mathbb{R}^n)} &\leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)}, \\ \|M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})\|_{L_{w^{\beta_2}}^{\beta_2}(\mathbb{R}^n)} &\leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

Now by Lemma 2, the Hölder's inequality and the above two inequalities we get

$$\begin{aligned} \|I_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_{wP}^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} |I_{\Omega, \alpha}^{(m)}(\vec{f})(x) w(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} [M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})(x) w(x)]^{\frac{p}{2}} [M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})(x) w(x)]^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\ &\leq \|M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})\|_{L_{w^{\beta_1}}^{\beta_1}(\mathbb{R}^n)}^{\frac{1}{2}} \|M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})\|_{L_{w^{\beta_2}}^{\beta_2}(\mathbb{R}^n)}^{\frac{1}{2}} \\ &\leq C \prod_{i=1}^m \|f_i\|_{L_{w^{p_i}}^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

This completes the proof of Theorem 4.

5. TWO-WEIGHTED ESTIMATES FOR ROUGH MULTILINEAR FRACTIONAL INTEGRAL OPERATORS AND ROUGH MULTI-SUBLINEAR FRACTIONAL MAXIMAL OPERATORS

For the proofs of Theorems 5 and 6, we need some lemmas.

**Lemma 7.** [12] *Let  $1 < p < q < \infty$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $0 \leq \alpha < n$ . Suppose that  $(u, v)$  is a pair of weights for which there exists  $r > 1$  such that, for every cube  $Q$  in  $\mathbb{R}^n$ ,*

$$|Q|^{\frac{1}{q} + \frac{\alpha}{n} - \frac{1}{p}} \left( \frac{1}{|Q|} \int_Q u(x) dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_Q v(x)^{r(1-p')} dx \right)^{\frac{1}{rp'}} \leq C.$$

Then for every  $f \in L_v^p(\mathbb{R}^n)$  it follows that

$$\left( \int_{\mathbb{R}^n} |M_\alpha f(x)|^q u(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{\frac{1}{p}}.$$

This lemma can be extended to the multilinear case as following.

**Lemma 8.** *Let  $0 < \alpha < mn$ ,  $1 < p_i < q_i < \infty$  for each  $i = 1, 2, \dots, m$ . Let  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $(u, v)$  be a pair of weights. If there exists  $r_i > 1$  such that, for every cube  $Q$  in  $\mathbb{R}^n$ ,*

$$|Q|^{\frac{1}{q_i} + \frac{\alpha}{mn} - \frac{1}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x) dx \right)^{\frac{1}{q_i}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1-p'_i)} dx \right)^{\frac{1}{r_i p'_i}} \leq C,$$

then for arbitrary  $f_i \in L_v^{p_i}(\mathbb{R}^n)$ , there is a constant  $C > 0$ , independent of  $f_i$ , such that

$$(5.1) \quad \left\| M_\alpha^{(m)}(\vec{f}) \right\|_{L_u^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_v^{p_i}(\mathbb{R}^n)}.$$

*Proof.* Since  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ , by the Hölder's inequality, Lemma 5 and Lemma 7, we have

$$\begin{aligned} \left\| M_\alpha^{(m)}(\vec{f}) \right\|_{L_u^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} \left| M_\alpha^{(m)}(\vec{f})(x) \right|^p u(x) dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} \left| \prod_{i=1}^m M_{\frac{\alpha}{m}} f_i(x) \right|^p u(x) dx \right)^{\frac{1}{p}} \\ &\leq C \prod_{i=1}^m \left( \int_{\mathbb{R}^n} |M_{\frac{\alpha}{m}} f_i(x)|^{q_i} u(x) dx \right)^{\frac{1}{q_i}} \\ &\leq C \prod_{i=1}^m \left( \int_{\mathbb{R}^n} |f_i(x)|^{p_i} v(x) dx \right)^{\frac{1}{p_i}}. \end{aligned}$$

This proves the inequality (5.1).  $\square$

**The proof of Theorem 5.** From the conditions of Theorem 5, one notes that  $|f_i|^{s'} \in L_{v^{s'}}^{\frac{p_i}{s'}}(\mathbb{R}^n)$ ,  $\frac{1}{s'} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ . Since  $s' < p_i < q_i < \infty$  for each  $i$ , then  $\frac{p_i}{s'} > 1$ . Lemma 1 and Lemma 8 imply that

$$\begin{aligned} \|M_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_u^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} |M_{\Omega, \alpha}^{(m)}(\vec{f})(x)|^p u(x) dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} |M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x)|^{\frac{p}{s'}} u(x) dx \right)^{\frac{1}{p}} \\ &= \|M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})\|_{L_u^{\frac{p}{s'}}(\mathbb{R}^n)}^{\frac{1}{s'}} \\ &\leq C \prod_{i=1}^m \|f_i\|_{L_v^{\frac{p_i}{s'}}(\mathbb{R}^n)}^{\frac{1}{s'}} \leq C \prod_{i=1}^m \|f_i\|_{L_v^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

This proves inequality (2.8).

**The proof of Theorem 6.** Under the conditions of Theorem 6, we can choose a positive number  $\epsilon$  such that

$$0 < \epsilon < \min \left\{ \alpha, \frac{mn}{s'} - \alpha, \frac{n}{p}, mn \left( \frac{1}{p_i} - \frac{1}{mp} \right), \frac{1}{pr'_i} \right\}.$$

Let  $\frac{1}{l_1} = \frac{1}{p} - \frac{\epsilon}{n}$ ,  $\frac{1}{l_2} = \frac{1}{p} + \frac{\epsilon}{n}$ , then  $\frac{p}{2l_1} + \frac{p}{2l_2} = 1$ . Applying Lemma 2 and the Hölder's inequality, we get

$$\begin{aligned} \|I_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_u^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} |I_{\Omega, \alpha}^{(m)}(\vec{f})(x)|^p u(x) dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} [M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})(x)]^{\frac{p}{2}} [M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})(x)]^{\frac{p}{2}} u(x) dx \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} [M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})(x)]^{l_1} u(x)^{\frac{l_1}{p}} dx \right)^{\frac{1}{2l_1}} \times \\ &\quad \left( \int_{\mathbb{R}^n} [M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})(x)]^{l_2} u(x)^{\frac{l_2}{p}} dx \right)^{\frac{1}{2l_2}} \\ &\leq C \|M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})\|_{L_{u^{\frac{l_1}{p}}}^{l_1}(\mathbb{R}^n)}^{\frac{1}{2}} \|M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})\|_{L_{u^{\frac{l_2}{p}}}^{l_2}(\mathbb{R}^n)}^{\frac{1}{2}}. \end{aligned}$$

Suppose that  $\frac{1}{g_i} = \frac{1}{mp} - \frac{\epsilon}{mn}$ ,  $\frac{1}{h_i} = \frac{1}{mp} + \frac{\epsilon}{mn}$ . Then due to the way we choose  $\epsilon$ , we have  $s' < p_i < g_i < \infty$ ,  $s' < p_i < h_i < \infty$ , and moreover

$$\frac{1}{l_1} = \frac{1}{g_1} + \frac{1}{g_2} + \dots + \frac{1}{g_m}, \quad \frac{1}{l_2} = \frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_m}.$$

Observe that  $1 < \frac{l_1}{p} < r_i$ , we have

$$\begin{aligned}
& |Q|^{\frac{s'}{g_i} + \frac{(\alpha+\epsilon)s'}{mn} - \frac{s'}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x)^{\frac{l_1}{p}} dx \right)^{\frac{s'}{g_i}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1-(\frac{p_i}{s'})')} dx \right)^{\frac{1}{r_i(\frac{p_i}{s'})'}} \\
& \leq |Q|^{\frac{s'}{mp} + \frac{\alpha s'}{mn} - \frac{s'}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x)^{r_i} dx \right)^{\frac{s'}{r_i mp}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1-(\frac{p_i}{s'})')} dx \right)^{\frac{1}{r_i(\frac{p_i}{s'})'}} \\
& \leq C.
\end{aligned}$$

Thus the pair of weights  $(u^{\frac{l_1}{p}}, v)$  satisfies condition (2.7) with  $s' < p_i < g_i < \infty$  and  $\alpha + \epsilon$ . Then, Theorem 5 implies that  $M_{\Omega, \alpha + \epsilon}^{(m)}$  is bounded from  $L_v^{p_1}(\mathbb{R}^n) \times L_v^{p_2}(\mathbb{R}^n) \times \dots \times L_v^{p_m}(\mathbb{R}^n)$  to  $L_{u^{\frac{l_1}{p}}}^{l_1}(\mathbb{R}^n)$ . On the other hand, we can see  $\frac{l_2}{p} < 1 < r_i$  and so

$$\begin{aligned}
& |Q|^{\frac{s'}{h_i} + \frac{(\alpha-\epsilon)s'}{mn} - \frac{s'}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x)^{\frac{l_2}{p}} dx \right)^{\frac{s'}{h_i}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1-(\frac{p_i}{s'})')} dx \right)^{\frac{1}{r_i(\frac{p_i}{s'})'}} \\
& \leq |Q|^{\frac{s'}{mp} + \frac{\alpha s'}{mn} - \frac{s'}{p_i}} \left( \frac{1}{|Q|} \int_Q u(x)^{r_i} dx \right)^{\frac{s'}{r_i mp}} \left( \frac{1}{|Q|} \int_Q v(x)^{r_i(1-(\frac{p_i}{s'})')} dx \right)^{\frac{1}{r_i(\frac{p_i}{s'})'}} \\
& \leq C.
\end{aligned}$$

In this case, the pair of weights  $(u^{\frac{l_2}{p}}, v)$  verifies condition (2.7) with  $s' < p_i < h_i < \infty$  and  $\alpha - \epsilon$ . Then, Theorem 5 implies that  $M_{\Omega, \alpha - \epsilon}^{(m)}$  is a bounded operator from  $L_v^{p_1}(\mathbb{R}^n) \times L_v^{p_2}(\mathbb{R}^n) \times \dots \times L_v^{p_m}(\mathbb{R}^n)$  to  $L_{u^{\frac{l_2}{p}}}^{l_2}(\mathbb{R}^n)$ .

Combining above estimates together, we get

$$\begin{aligned}
\|I_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_u^p(\mathbb{R}^n)} & \leq C \|M_{\Omega, \alpha + \epsilon}^{(m)}(\vec{f})\|_{L_{u^{\frac{l_1}{p}}}^{l_1}(\mathbb{R}^n)}^{\frac{1}{2}} \|M_{\Omega, \alpha - \epsilon}^{(m)}(\vec{f})\|_{L_{u^{\frac{l_2}{p}}}^{l_2}(\mathbb{R}^n)}^{\frac{1}{2}} \\
& \leq \prod_{i=1}^m \|f_i\|_{L_v^{p_i}(\mathbb{R}^n)}.
\end{aligned}$$

This is desired inequality (2.9) of Theorem 6. The proof of Theorem 6 is completed.

## 6. TWO-WEIGHTED WEAK-TYPE ESTIMATES FOR ROUGH MULTI-SUBLINEAR FRACTIONAL MAXIMAL OPERATORS

In this section, we give some two-weighted weak-type inequalities for multi-sublinear fractional maximal operator  $M_{\alpha}^{(m)}$  and rough multi-sublinear fractional maximal operator  $M_{\Omega, \alpha}^{(m)}$ , respectively.



**Lemma 9.** [5] *Let  $1 \leq p \leq q < \infty$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $0 \leq \alpha < n$ . Let  $(u, v)$  be a pair of weights in  $A_{p,q}^\alpha$ . Then for every  $\lambda > 0$ ,*

$$u(\{x \in \mathbb{R}^n : |M_\alpha f(x)| > \lambda\}) \leq \frac{C}{\lambda^q} \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{\frac{q}{p}}.$$

**Lemma 10.** *Let  $0 < \alpha < mn$ ,  $1 \leq p_i \leq q_i < \infty$  for each  $i = 1, 2, \dots, m$ . Let  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $(u, v)$  be a pair of weights. If  $(u, v) \in \bigcap_{i=1}^m A_{p_i, q_i}^{\frac{\alpha}{m}}$ , then for every  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$ , there is a constant  $C$ , independent of  $f_i$ , such that*

$$(6.1) \quad \left\| M_\alpha^{(m)}(\vec{f}) \right\|_{L_{u, \infty}^{p, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}.$$

*Proof.* For any fixed  $\lambda > 0$ , we denote by

$$\mu_k = \lambda^{\frac{p}{q_k}} \|f_k\|_{L_{v^k}^{p_k}(\mathbb{R}^n)} \left( \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)} \right)^{-\frac{p}{q_k}}, \quad k = 1, 2, \dots, m-1.$$

Since  $1 \leq p_i \leq q_i < \infty$  for every  $i$  and  $(u, v) \in \bigcap_{i=1}^m A_{p_i, q_i}^{\frac{\alpha}{m}}$ , then Lemma 5 and Lemma 9 give that

$$\begin{aligned} & u\left(\left\{x: \left|M_\alpha^{(m)}(\vec{f})\right| > \lambda\right\}\right) \\ & \leq u\left(\left\{x: \left|\prod_{i=1}^m M_{\frac{\alpha}{m}} f_i(x)\right| > \frac{\lambda}{C}\right\}\right) \\ & \leq \sum_{k=1}^{m-1} u\left(\left\{x: \left|M_{\frac{\alpha}{m}} f_k(x)\right| > \mu_k\right\}\right) + u\left(\left\{x: \left|M_{\frac{\alpha}{m}} f_m(x)\right| > \frac{\lambda}{C} \prod_{k=1}^{m-1} \mu_k^{-1}\right\}\right) \\ & \leq \sum_{k=1}^{m-1} C \left(\mu_k^{-1} \|f_k\|_{L_{v^k}^{p_k}(\mathbb{R}^n)}\right)^{q_k} + C \left(\lambda^{-1} \prod_{k=1}^{m-1} \mu_k \|f_m\|_{L_{v^m}^{p_m}(\mathbb{R}^n)}\right)^{q_m} \\ & \leq C \left(\frac{1}{\lambda} \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}\right)^p. \end{aligned}$$

This yields desired inequality (6.1).  $\square$

**Theorem 8.** *Let  $0 < \alpha < mn$ ,  $1 \leq s' < \frac{mn}{\alpha}$ , and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(S^{mn-1})$  ( $s > 1$ ),  $\frac{1}{s} + \frac{1}{s'} = 1$ . Assume that  $(u, v)$  is a pair of weights,  $s' \leq p_i \leq q_i < \infty$  for each  $i = 1, 2, \dots, m$ .  $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ ,  $\frac{1}{p_i} + \frac{1}{p'_i} = 1$ . If  $(u, v) \in \bigcap_{i=1}^m A_{\frac{p_i}{s'}, q_i}^{\frac{\alpha s'}{s}}$ , then for every  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$ , there is a constant  $C$ , independent of  $f_i$ , such that*

$$(6.2) \quad \left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_{u, \infty}^{p, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}.$$

*Proof.* Since  $s' \leq p_i \leq q_i < \infty$  for each  $i$  and  $(u, v) \in \bigcap_{i=1}^m A_{\frac{p_i}{s'}, \frac{q_i}{s'}}^{\frac{\alpha s'}{m}}$ , we can see by Lemma 1 and Lemma 10 that

$$\begin{aligned}
\left\| M_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right\|_{L_u^{p, \infty}(\mathbb{R}^n)} &= \sup_{\lambda > 0} \lambda \left( u \left( \left\{ x : \left| M_{\Omega, \alpha}^{(m)} \left( \vec{f} \right) \right| > \lambda \right\} \right) \right)^{\frac{1}{p}} \\
&\leq C \sup_{\lambda > 0} \lambda \left( u \left( \left\{ x : \left| \left[ M_{\alpha s'}^{(m)} \left( |f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'} \right) (x) \right] \right| > \lambda^{s'} \right\} \right) \right)^{\frac{1}{p}} \\
&\leq C \sup_{\theta > 0} \theta^{\frac{1}{s'}} \left( u \left( \left\{ x : \left| \left[ M_{\alpha s'}^{(m)} \left( |f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'} \right) (x) \right] \right| > \theta \right\} \right) \right)^{\frac{1}{p}} \\
&\leq C \left\| M_{\alpha s'}^{(m)} \left( |f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'} \right) \right\|_{L_u^{\frac{p}{s'}, \infty}(\mathbb{R}^n)}^{\frac{1}{s'}} \\
&\leq C \prod_{i=1}^m \left\| |f_i|^{s'} \right\|_{L_v^{\frac{p_i}{s'}}(\mathbb{R}^n)}^{\frac{1}{s'}} \leq C \prod_{i=1}^m \|f_i\|_{L_v^{p_i}(\mathbb{R}^n)}.
\end{aligned}$$

This is desired inequality (6.2). We complete the proof of Theorem 8.  $\square$

**Remark 3.** We note that if a pair weights  $(u, v)$  satisfies (2.7) with  $r_i > 1$ , then  $(u, v) \in \bigcap_{i=1}^m A_{\frac{p_i}{s'}, \frac{q_i}{s'}}^{\frac{\alpha s'}{m}}$ . This result says that, by  $A_{\frac{p_i}{s'}, \frac{q_i}{s'}}^{\frac{\alpha s'}{m}}$ , which is weaker than the condition (2.7), the operator  $M_{\Omega, \alpha}^{(m)}$  turns out to be of weak type.

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